Kuwait University	Math. 101	Date	: October 14th,1999.
Dept.of Maths.&Comp. Sci.	First Exam.	Duration	: 75 minutes.

Calculators, Mobile Telephones and Pagers are not allowed.
Answer all the following questions
(Each question is weighted 5 points)

1. (a) Evaluate the following limit, if it exists: $\lim\limits_{x\to 3} \dfrac{\sin(\pi(x-3))}{x^2-9}$.

   (b) Find $f'(x)$, where

$$f(x) = \left(2x^3+5\right)^5 + \tan\left(\frac{x-3}{x+3}\right).$$

2. Find the vertical and horizontal asymptotes, if any, of the graph of the function f, where

$$f(x) = \frac{\sqrt{x^4+2x^2}}{x^2+2x}.$$

3. Classify the discontinuities of f as removable, jump, or infinite, where

$$f(x) = \begin{cases} x^2+1 & \text{if } x < 0, \\ \dfrac{x-4}{x^2-5x+4} & \text{if } x \geqslant 0. \end{cases}$$

4. (a) Does the equation $x^3 - x - 1 = 0$ have a solution in the interval $[1,2]$? Explain.

   (b) Let

$$f(x) = x(x-1)^{\frac{2}{3}}.$$

   Does the graph of the function f have a cusp? Explain.

5. Let

$$f(x) = \sqrt{x} + 5.$$

   (a) Use the definition of the derivative to find $f'(1)$.

   (b) Find an equation of the tangent line to the graph of f at the point $P(1,6)$.

Good Luck

Q1

Part (a)

$$\lim_{x\to3}\frac{\sin(\pi(x-3))}{(x-3)(x+3)} = \pi\lim_{\pi(x-3)\to0}\frac{\sin(\pi(x-3))}{\pi(x-3)} \quad \therefore \lim_{x\to3}\left(\frac{1}{x+3}\right)=\frac{\pi}{6}$$

Part (b)

$$f'(x)=5(2x^3+4)^4(6x^2)+\sec^2\left(\frac{x-3}{x+3}\right)\frac{(x+3)(1)-(x-3)(1)}{(x+3)^2}$$

$$=30x^2(2x^3+5)^4+\frac{6}{(x+3)^2}\sec^2\left(\frac{x-3}{x+3}\right)$$

Q2

(I) V.A:

$$x^2+2x=0 \implies x(x+2)=0 \implies x=0 \text{ or } x=-2$$

at $x=0$

$$\lim_{x\to0}f(x)=\lim_{x\to0}\frac{|x|\sqrt{x^2+2}}{x(x+2)}=\text{DNE, for}$$

$$\lim_{x\to0^+}\frac{|x|\sqrt{x^2+2}}{x(x+2)}=\frac{\sqrt2}{2};\ \lim_{x\to0^-}\frac{|x|\sqrt{x^2+2}}{x(x+2)}=-\frac{\sqrt2}{2} \implies \text{No V.A @ } x=0$$

at $x=-2$

$$\left.\begin{aligned}\lim_{x\to-2^+}\frac{|x|\sqrt{x^2+2}}{x(x+2)}&=-\infty\\ \lim_{x\to-2^-}\frac{|x|\sqrt{x^2+2}}{x(x+2)}&=+\infty\end{aligned}\right\} \implies \boxed{x=-2} \text{ is V.A.}$$

(II) H.A.

$$\lim_{x\to\infty}f(x)=\lim_{x\to\infty}\frac{|x|\sqrt{x^2+2}}{x(x+2)}=\lim_{x\to\infty}\frac{|x|^2\sqrt{1+\frac{2}{x}}}{x^2(1+2/x)}=1 \implies \boxed{y=1} \text{ is H.A.}$$

$$\lim_{x\to-\infty}f(x)= \qquad \implies \boxed{y=1} \text{ is H.A}$$

Q3

$f(0) = -1$

$\lim_{x\to 0^+} f(x) = \lim_{x\to 0} \frac{x-4}{(x-4)(x-1)} = -1$

$\lim_{x\to 0^-} f(x) = 1$

} $f(x)$ has jump discon. @ $x=0$

@ $x=1$

$f(1)$ is UD.

$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} \frac{(x-4)}{(x-4)(x-1)} = +\infty$, $\lim_{x\to 1^-} f(x) = -\infty$ } $f(x)$ has infinite disc. @ $x=1$

@ $x=4$

$f(4)$ is UD.

$\lim_{x\to 4} \frac{x-4}{(x-4)(x-1)} = 1/3$ } f has removable disc. @ $x=4$

Q4

Part(a) let $f(x) = x^3 - x - 1$ f is Cont. $\forall x \in R$ (Poly. function)

f is Conti on $[1,2]$

$f(1) = -1 < 0$, $f(2) = 5 > 0$, therfore, for the I.V.T.

f has a zero in $[1,2]$, therfore The eq. $x^3 - x - 1 = 0$ has a real Solution in $[1,2]$.

Part(b)

$f'(x) = 2/3\, x\, (x-1)^{-1/3} + (x-1)^{2/3} = \frac{5x-3}{3(x-1)^{1/3}}$

f is Cont. @ $x=1$

$\lim_{x\to 1^+} f'(x) = \lim_{x\to 1^+} \frac{5x-3}{3(x-1)^{1/3}} = +\infty$, $\lim_{x\to 1^-} f'(x) = -\infty$

~~therefore~~ the graph of f has a Cusp @ $x=1$

Q_5

Part (a)

$$f'(1) = \lim_{x\to 1} \frac{f(x)-f(1)}{x-1} = \lim_{h\to 0} \frac{f(1+h)-f(1)}{h}$$

$$= \lim_{x\to 1} \frac{(\sqrt{x}+5)-(\sqrt{1}+5)}{(x-1)} = \lim_{x\to 1} \frac{\sqrt{x}-1}{x-1} = \lim_{x\to 1} \frac{(\sqrt{x}-1)}{(\sqrt{x}-1)(\sqrt{x}+1)}$$

$$= 1/2$$

Part (b)

$m_a = f'(1) = 1/2$

Eq. of tangent line @ $P(1,6)$:

$y - y_1 = m_a(x - x_1)$

$y - 6 = (1/2)(x-1)$

$2y = x + 11$